(10%) 1. Find the general solution of the differential equation

$$y' + \frac{1}{x}y = 3e^{-x^2}$$

(15%) 2. Find the general solution of the differential equation

$$y'' + 9y = 12\sec(3x)$$

(10%) 3. Let

$$y'' + 2y' + 8y = 0; \quad y(0) = 2, \ y'(0) = 0.$$

$$y_1(x) = e^{-x}\cos(\sqrt{7}x), \ y_2(x) = e^{-x}\sin(\sqrt{7}x).$$

(a) verify that y_1 and y_2 are solutions of the differential equation,

(b) show that y_1 and y_2 are linearly independent solutions of the differential equation,

(c) write the general solution of the differential equation, and

(d) find the solution of the initial value problem.

(15%) 4. Find the initial value problem using the Laplace transform

$$y'' + 4y' + 3y = e^t; \quad y(0) = 0, \ y'(0) = 2.$$

(12%) 5. Find the eigenvalues and a set of eigenvectors of the real symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & -2 \\ -1 & -2 & 2 \end{bmatrix}$$

Verify that its eigenvectors are mutually orthogonal.

(13%) 6. Find the condition(s) on α such that the simultaneous equations

$$x_1 + \alpha x_2 = 1,$$
$$x_1 - x_2 + 3x_3 = -1,$$
$$2x_1 - 2x_2 + \alpha x_3 = -2$$

Have (a) exact one solution, (b) no solution, or (c) an infinite number of solutions; given all solutions where they exist.

(12%) 7. Find the Fourier series of the 2-periodic function

$$f(x) = 1 - x^2 \ \text{ if } -1 < x < 1$$

(13%) 8. Let a and b be real numbers and $k > 0$. Consider the partial differential equation $w_t = k(w_{xx} + aw_x + bw)$ Determine α and β so that, if $w(x,t) = e^{\alpha x + \beta t}u(x,t)$ then u satisfies the standard heat equation ($u_t = ku_{xx}$)